Tail behaviour of β-TARCH models

Péter Elek[*,a,b,1], László Márkus[a]

[a]*Department of Probability Theory and Statistics, Eötvös Loránd University, Hungary*
H-1117 Budapest Pázmány Péter sétány 1/c.
[b]*Economic Policy Department, Ministry of Finance, Hungary*

Abstract

It is now common knowledge that the simple quadratic ARCH process has regularly varying tail even when generated by a normally distributed noise, and the tail behaviour is well-understood under more general conditions as well. Much less studied is the case of β-ARCH-type processes, i.e. when the conditional variance is a 2β-power function with $0 < \beta < 1$ and when light tails do not produce polynomially decaying tails. Being a little more general and allowing for asymmetry, we consider threshold β-ARCH models, driven by noises with Weibull-like tails. (Special cases include the Gaussian or the Laplace distributions.) We show that the generated process has approximately Weibull like tail, too, albeit with different exponent: $1-\beta$ times that of the noise, in the sense that the tail can be bounded from both sides by Weibull distributions of this exponent but slightly different constants. The proof is based on taking an appropriate auxiliary sequence and then applying the general result of Klüppelberg and Lindner (2005) for the tail of infinite MA sequences with light-tailed innovations.

Key words: β-TARCH model, conditional heteroscedasticity, extreme value theory, tail behaviour

[*]Corresponding author
Email addresses: elekpeti@cs.elte.hu (Péter Elek), markus@cs.elte.hu (László Márkus)
[1]Phone number: +36-30-474-3831

1. Introduction

In this paper we examine the tail behaviour of the stationary distribution of certain ARCH-type models defined by the equation

$$X_t = \left(\omega + \alpha_+ \left(X_{t-1}^+\right)^{2\beta} + \alpha_- \left(X_{t-1}^-\right)^{2\beta}\right)^{1/2} Z_t, \tag{1}$$

where we apply the usual notations $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$. The model parameters ω, α_+ and α_- satisfy $\min(\alpha_+, \alpha_-) \geq 0$, $\max(\alpha_+, \alpha_-) > 0$ and $\omega > 0$, and Z_t is an i.i.d. sequence with zero mean and finite variance.

An important feature of this process is that – if the autocorrelation function is defined at all – it is an uncorrelated but not an independent sequence because its conditional variance is changing over time as a function of the lagged values (conditional heteroscedasticity). If $\beta = 1$ and $\alpha_+ = \alpha_- > 0$, we obtain the well-known ARCH (autoregressive conditionally heteroscedastic) model (Engle, 1982), where the conditional variance has a quadratic functional form. Since this process can reproduce the stylised facts (e.g. uncorrelatedness, conditional heteroscedasticity, nonnormality) of financial time series in an easily estimable way, it has become a basic tool in financial econometrics in the past two decades, and gave rise to various generalisations. For instance, in order to model the fact that the variance of stock returns responds more strongly to negative shocks than to positive ones, Glosten et al. (1993) defined the TARCH (threshold ARCH) process by allowing $\alpha_+ \neq \alpha_-$ in the equation with $\beta = 1$. (Hence $\alpha_+ < \alpha_-$ generally holds in financial applications.) For a broad overview of the various generalisations of ARCH models and some of their properties we refer the reader to Terasvirta (2009).

Due to the popularity of the quadratic ARCH models in finance, their probabilistic properties are quite much studied and well understood. It is a well-known fact for $\beta = 1$ (see e.g. Embrechts et al. (1997)) that not all choices of $(\omega, \alpha_+, \alpha_-)$ and of the distribution of Z_t permit a stationary solution of equation (1). For instance, if Z_t is normally distributed, the quadratic ARCH model (i.e. the case $\alpha_+ = \alpha_-$) has a stationary solution if and only if $\alpha_+ = \alpha_- < 2\exp(\delta) \approx 3.562$, where δ is the Euler-constant. (A different choice for the noise distribution yields a different domain of stationarity.)

Also, much is known about the tail behaviour of the stationary distribution if $\beta = 1$. It was proven two decades ago (Goldie, 1991) that the simple ARCH process has regularly varying tail (roughly speaking: polynomially decaying tail) even when Z_t is normally distributed. This phenomenon is often summarised as: "light-tailed input can cause heavy-tailed output". More generally, Borkovec and Klüppelberg (2001) proved that the AR(1) model driven by a quadratic ARCH(1) innovation has regularly varying tail for a very general class of noise distributions. Using the concepts of extreme value theory (EVT) it follows that the stationary distribution of quadratic ARCH processes belongs to the maximum domain of attraction of the Frechet extreme value distribution or, equivalently, their tail can be approximated by a generalised Pareto distribution (GPD) with shape parameter $\xi > 0$. (For an introduction into EVT we refer the reader to Embrechts et al. (1997).)

The $0 < \beta < 1$ case – where the conditional variance is increasing slower than a quadratic function of the lagged values – is very different from the usual $\beta = 1$ parameter choice, and is much less studied in the literature. The model may then be called the β-TARCH process and was analysed e.g. by Guegan and Diebolt (1994). It follows relatively easily from the drift condition for Markov chains (Meyn and Tweedie, 1993) that in the $0 < \beta < 1$ case the X_t process defined by (1) is stationary irrespective of the choice for the parameters and for the distribution of Z_t (provided that the latter has a finite second moment). Moreover, if the *m*th moment of Z_t is finite, the *m*th moment of the stationary distribution of X_t will be finite, too (see Guegan and Diebolt (1994), or in a more general setting Elek and Márkus (2008)). Hence, if all moments of Z_t is finite and its distribution has infinite support, the distribution of X_t may only belong to the maximum domain of attraction of the Gumbel law and, equivalently, the shape parameter of the GPD fitted to it may only be zero – if the distribution belongs to the maximum domain of attraction of an extreme value law at all.

This result already yields that the β-TARCH model is lighter tailed than the usual, quadratic specification: for light-tailed Z_t noises the tail of X_t decays faster to zero than a polynomial function. The finding, however, does not determine the exact tail behaviour: the maximum domain of attraction of the Gumbel law contains many different types of distributions (e.g. normally, exponentially or lognormally decaying ones). In this paper we give a more precise estimate for the tail decay by showing that X_t has approximately Weibull-like tail provided that Z_t has a Weibull-like distribution. Our research is motivated by the fact that β-TARCH models proved useful to model conditional heteroscedasticity in areas where the quadratic ARCH model was considered too heavy-tailed, such as in the analysis of water discharge series of rivers with large catchments. (See e.g. Elek and Márkus (2008) or in a broader context Szilágyi et al. (2006).)

Throughout the paper we will use the notations $\bar{F}_X(u) = 1 - F_X(u)$ for the survival function and $f_X(u)$ for the density function of the random variable X.

2. Tail behaviour

To examine the tail behaviour of X_t let us introduce an assumption on the tail of Z_t :

Assumption 1. *Z_t is an i.i.d. sequence with an absolutely continuous probability distribution. Moreover, there exist $u_0 > 0$, $\gamma > 0$, $\kappa > 0$, $K_1 > 0$ and K_2 such that its probability density satisfies*

$$f_{Z_t}(u) = K_1 |u|^{K_2} \exp\left(-\kappa |u|^{\gamma}\right) \tag{2}$$

for every $|u| > u_0$.

According to this assumption, Z_t is symmetric and has a Weibull-like tail with exponent γ. The Gaussian $(\gamma = 2)$ or the Laplace $(\gamma = 1)$ distributions are obtained as special cases.

Guegan and Diebolt (1994) showed under the assumption $\min(\alpha_+, \alpha_-) > 0$ that if $\beta > (\gamma - 1)/\gamma$, X_t has no exponential moment (i.e. it is heavier tailed than the exponential distribution) while if $\beta < (\gamma - 1)/\gamma$, X_t has a moment generating function defined around the neighbourhood of zero. This finding already suggests that X_t may possess (approximately) a Weibull-like tail with exponent $\gamma(1 - \beta)$. Assuming a normally distributed noise (i.e. $\gamma = 2$), $\alpha_+ = \alpha_-$ and $1/2 < \beta < 1$, Robert (2000) argued that this is indeed the case: under his assumptions X_t has Weibull-like tail with exponent $2(1-\beta)$. Although the proof of his conjectures seems to be incomplete,[2] some of his ideas are useful to prove that X_t has approximately Weibull-like tail even if we consider the more general case, i.e. $\alpha_+ \neq \alpha_-$, $\gamma \neq 2$ and $0 < \beta \leq 1/2$.

Theorem 1. *Assume that X_t satisfies equation* (1)*, Assumption 1 holds, and* $\omega > 0$, $\min(\alpha_+, \alpha_-) > 0$, $0 < \beta < 1$. *Then, using the notation* $\alpha = \max(\alpha_+, \alpha_-)$,

$$\exp\left(-\frac{\alpha^{-\gamma/2}\kappa\gamma\beta^{-\frac{\beta}{1-\beta}}}{2}u^{\gamma(1-\beta)} + O\left(u^{\gamma(1-\beta)/2}\right)\right) \leq \bar{F}_{X_t}(u)$$
$$\leq \exp\left(-\frac{(\alpha+\omega)^{-\gamma/2}\kappa\gamma\beta^{-\frac{\beta}{1-\beta}}}{2}u^{\gamma(1-\beta)} + O\left(u^{\gamma(1-\beta)/2}\right)\right). \quad (3)$$

Clearly, since $(K-\epsilon)z < Kz + O\left(z^{1/2}\right) < (K+\epsilon)z$ as $z \to \infty$ for any $K > 0$ and $\epsilon > 0$, the exponent and the bounds for the multiplier of $u^{\gamma(1-\beta)}$ are exact in the above given Weibull like approximation, but (even if the constants were the same in the upper and lower bound) there still remains room for a deviation from an asymptotically exact Weibull limit distribution.

Proof. We may assume without loss of generality that $\alpha = \alpha_+ \geq \alpha_-$. Let $Y_t = \log(X_t^2)$, $U_{t,1} = \log\left(\alpha_+ Z_t^2\right)$, and $U_{t,2} = \log\left(\alpha_- Z_t^2\right)$. Furthermore, let us introduce the functions

$$h_1(y) = \log(\omega/\alpha_+ + \exp(\beta y)),$$
$$h_2(y) = \log(\omega/\alpha_- + \exp(\beta y))$$

and the random variables $V_{t,i} = h_i(Y_{t-1}) - \beta Y_{t-1}$ $(i = 1, 2)$. Then

$$Y_t = h_1(Y_{t-1}) + U_{t,1} = \beta Y_{t-1} + U_{t,1} + V_{t,1} \qquad \text{if} \quad Z_{t-1} > 0,$$
$$Y_t = h_2(Y_{t-1}) + U_{t,2} = \beta Y_{t-1} + U_{t,2} + V_{t,2} \qquad \text{if} \quad Z_{t-1} \leq 0.$$

[2]He derives a functional equation for the logarithm of the moment generating function $L_Y(s) = E(\exp(sX_t))$ of $Y_t = \log X_t^2$ and estimates the tail of Y_t based on the behaviour of $L_Y(s)$ around ∞. During the calculations he assumes (see Appendix 1 of his paper) that if a function g satisfies $g(x) - g(\alpha x) = O(1/x)$ as $x \to \infty$, then $g(x) = O(1/x)$. However, this is not the case: if e.g. $g(x) = \sin(2\pi \log x / \log\alpha)$ then $g(x) - g(\alpha x) = 0$.

Since $h_i(y) \geq \beta y \ (i=1,2)$, $V_{t,i} \geq 0$ a.s. Moreover, since Z_t is a symmetrically distributed i.i.d. sequence, Y_t can be written as

$$Y_t = \beta Y_{t-1} + U_t + V_t,$$

where $U_t = U_{t,1}\chi_{\{Z_{t-1}>0\}} + U_{t,1}\chi_{\{Z_{t-1}\leq 0\}}$ is an independent 1/2-1/2 mixture of $U_{t,1}$ and $U_{t,2}$, and is itself and i.i.d. process. Similarly $V_t = V_{t,1}\chi_{\{Z_{t-1}>0\}} + V_{t,1}\chi_{\{Z_{t-1}\leq 0\}}$ is an independent 1/2-1/2 mixture of $V_{t,1}$ and $V_{t,2}$.

Let us introduce the auxiliary autoregressive sequence

$$Y_t^* = \beta Y_{t-1}^* + U_t = \sum_{i=0}^{\infty} \beta^i U_{t-i}.$$

As we use $\log$ in the definition of U_t, the convergence of the sum cannot be taken for granted. However, the distribution of Z_t is absolutely continuous (at zero as well) and it has finite variance, hence $E\left(U_t^2\right) < \infty$, therefore Y_t^* exists and $E\left(Y_t^*\right)^2 < \infty$. It is clear that $Y_t^* \leq Y_t$, so by examining the tail behaviour of Y_t^* we obtain a lower bound for the tail of Y_t as well.

To determine the tail of Y_t^*, we will apply the framework of Klüppelberg and Lindner (2005) who examined the tail behaviour of linear moving average processes with increments lighter tailed than the exponential distribution. Let $\sum_{-\infty}^{\infty} c_i W_{t-i}$ be the examined process and assume that the probability density of the i.i.d. sequence W_t satisfies

$$f(u) = \nu(u)\exp\left(-\psi(u)\right), \quad u \geq u_0 \tag{4}$$

for some u_0, and $\psi(u)$ is C^2, $\psi'(u_0) = 0$, $\psi'(\infty) = \infty$ and ψ'' is strictly positive on $[u_0, \infty]$. Moreover, using the notation $\phi = 1/\sqrt{\psi''}$, we assume that the function ν is flat for ϕ, i.e.

$$\lim_{u\to\infty} \frac{\nu\left(u + x\phi\left(u\right)\right)}{\nu\left(u\right)} = 1$$

uniformly on bounded x-intervals, and also that ϕ is self-neglecting, i.e. flat for itself. (Roughly speaking, these assumptions require that $\psi(u)$ should be strictly convex and $\nu(u)$ should behave approximately as a constant as $u \to \infty$.)

Furthermore, following the original notations of Klüppelberg and Lindner (2005), define $q(\tau) = \psi'^{-1}(\tau)$ and

$$Q(\tau) = \sum_{i=-\infty}^{\infty} c_i q(c_i \tau), \qquad \sigma_i^2(\tau) = c_i^2 q'(c_i \tau), \qquad \sigma_\infty^2(\tau) = \sum_{i=-\infty}^{\infty} \sigma_i^2(\tau).$$

It follows from the conditions that Q is a strictly increasing function. Then, provided that c_i is a summable sequence of non-negative real numbers, not all zero, and assuming that the two conditions below hold:

$$\lim_{m\to\infty} \limsup_{\tau\to\infty} \frac{\sum_{|j|>m} \sigma_j^2(\tau)}{\sigma_\infty^2(\tau)} = 0, \tag{5}$$

$$\lim_{m\to\infty} \limsup_{\tau\to\infty} \frac{\sum_{|j|>m} \sigma_j(\tau)}{\sigma_\infty^2(\tau)} = 0, \tag{6}$$

the following theorem is true:

Theorem 2. *(Klüppelberg and Lindner, 2005) Under the above conditions, as $u \to \infty$,*

$$P\left(\sum_{i=-\infty}^{\infty} c_i W_{t-i} > u\right) \sim \frac{1/\sqrt{2\pi}}{Q^{-1}(u)\,\sigma_\infty(Q^{-1}(u))} \exp\left(-\int_{u_0\sum c_i}^{u} \left(Q^{-1}(v) + \rho\left(Q^{-1}(v)\right)\right) dv\right)$$

where $\rho(\tau) = o(1/\sigma_\infty(\tau))$. It is also true that $1/\sigma_\infty(\tau) = o(\tau)$ so the first term in the integral is the leading term.

In our case, this theorem will be used with the choice $U_t = W_t$ so the conditions of the theorem should be checked first. Here,

$$f_{U_t}(u) = \frac{1}{2}\left(K_1 \exp(K_2 u) \exp\left(-\kappa\alpha_+^{-\gamma/2} e^{\frac{\gamma u}{2}}\right)\right) + \frac{1}{2}\left(K_3 \exp(K_4 u) \exp\left(-\kappa\alpha_-^{-\gamma/2} e^{\frac{\gamma u}{2}}\right)\right) = \nu(u)\exp(-\psi(u))$$

with appropriate constants $K_1 > 0$, $K_3 > 0$ and K_2, K_4. To satisfy the necessary assumptions with $u_0 = 0$ in (4), $\psi(u)$ can be defined as

$$\psi(u) = \kappa\alpha^{-\gamma/2} e^{\frac{\gamma u}{2}} - e\kappa\alpha^{-\gamma/2}/2 \qquad \text{if} \quad u \geq 2/\gamma,$$

$$\psi(u) = e\kappa\alpha^{-\gamma/2} (\gamma/2)^2 u^2/2 \qquad \text{if} \quad u < 2/\gamma.$$

Then it is a matter of routine to check that the resulting $\nu(u)$ function is flat for $\phi(u)$ and that $\phi(u)$ is self-neglecting (see also Example 2.4. (c) in Klüppelberg and Lindner

(2005)), so the tail of Y_t^* can in principle be approximated using $c_i = \beta^i$ for $i \geq 0$ and $c_i = 0$ for $i < 0$. (Conditions (5)-(6) will be checked, see below.) Using the notation $\tau_0 = e\kappa\alpha^{-\gamma/2}\gamma/2$, we obtain

$$\begin{aligned} \psi'(u) &= \kappa\alpha^{-\gamma/2}(\gamma/2)\, e^{\frac{\gamma u}{2}} = e^{-1}\tau_0 e^{\frac{\gamma u}{2}} && \text{if} \quad u \geq 2/\gamma, \\ \psi'(u) &= e\kappa\alpha^{-\gamma/2}(\gamma/2)^2 u = \tau_0(\gamma/2)\, u && \text{if} \quad u < 2/\gamma \end{aligned}$$

and hence

$$\begin{aligned} q(\tau) &= 2\gamma^{-1}\log(e\tau/\tau_0) = 2\gamma^{-1}(\log\tau - \log\tau_0 + 1) && \text{if} \quad \tau \geq \tau_0, \\ q(\tau) &= 2\gamma^{-1}\tau/\tau_0 && \text{if} \quad \tau < \tau_0. \end{aligned}$$

Then

$$\begin{aligned} Q(\tau) = \sum_{j=0}^{\infty}\beta^j q\left(\beta^j\tau\right) &= 2\gamma^{-1}\sum_{j=0}^{\infty}\beta^j\log\left(\beta^j e\tau/\tau_0\right) \\ &\quad + 2\gamma^{-1}\sum_{j:\ \beta^j\tau<\tau_0}\beta^j\left(\beta^j\tau/\tau_0 - \log\left(\beta^j e\tau/\tau_0\right)\right). \end{aligned} \tag{7}$$

For any $0 < \theta < 1$ (we shall choose θ appropriately from this range later), the sum in the second term can be written as

$$\begin{aligned} &\sum_{j:\ \beta^j\tau<\tau_0}\beta^j\left(\beta^j\tau/\tau_0 - \log\left(\beta^j e\tau/\tau_0\right)\right) \\ &= (e\tau/\tau_0)^{-\theta}\sum_{j:\ \beta^j\tau<\tau_0}\left(\beta^{1-\theta}\right)^j\left(e^\theta\left(\beta^j\tau/\tau_0\right)^{1+\theta} - \left(\beta^j e\tau/\tau_0\right)^\theta\log\left(\beta^j e\tau/\tau_0\right)\right). \end{aligned} \tag{8}$$

For $0 < \theta < 1$ the function $g(x) = x^\theta\log x$ is bounded on $(0, e]$, hence both $\left(\beta^j e\tau/\tau_0\right)^\theta\log\left(\beta^j e\tau/\tau_0\right)$ and $e^\theta\left(\beta^j\tau/\tau_0\right)^{1+\theta}$ are bounded if $0 < \beta^j\tau/\tau_0 < 1$. This way we have for $\tau \to \infty$

$$\begin{aligned} &\sum_{j:\ \beta^j\tau<\tau_0}\left(\beta^{1-\theta}\right)^j\left(e^\theta\left(\beta^j\tau/\tau_0\right)^{1+\theta} - \left(\beta^j e\tau/\tau_0\right)^\theta\log\left(\beta^j e\tau/\tau_0\right)\right) \\ &\qquad = O(1)\sum_{j:\ \beta^j\tau<\tau_0}\left(\beta^{1-\theta}\right)^j = o(1) \end{aligned}$$

because $\tau \to \infty$ implies that $j \to \infty$ in the summation condition.

Therefore the sum in (8) (and so the second term in (7)) is $o\left(\tau^{-\theta}\right)$. As a consequence, using the notations

$$A = 2\gamma^{-1}\left(1-\beta\right)^{-1} \quad \text{and} \quad B = \beta\left(1-\beta\right)^{-1}\log\beta + 1 - \log\tau_0,$$

we obtain for $\tau \to \infty$

$$\begin{aligned} Q\left(\tau\right) &= 2\gamma^{-1}\sum_{j=0}^{\infty}\beta^j\left(\log\left(\beta^j\tau\right) + 1 - \log\tau_0\right) + o\left(\tau^{-\theta}\right) \\ &= 2\gamma^{-1}\left(1-\beta\right)^{-1}\Big(\log\tau + \beta\left(1-\beta\right)^{-1}\log\beta + 1 - \log\tau_0\Big) + o\left(\tau^{-\theta}\right) \\ &= A\left(\log\tau + B\right) + o\left(\tau^{-\theta}\right). \end{aligned}$$

Trivially, $Q^{-1}\left(u\right) \to \infty$ and hence

$$\exp\left(-o\left(\left(Q^{-1}\left(u\right)\right)^{-\theta}\right)\right) = 1 + o\left(\left(Q^{-1}\left(u\right)\right)^{-\theta}\right) = 1 + o\left(1\right) \tag{9}$$

as $u \to \infty$, therefore

$$Q^{-1}\left(u\right) = \exp\left(A^{-1}u - B\right)\exp\left(-o\left(\left(Q^{-1}\left(u\right)\right)^{-\theta}\right)\right) = \exp\left(A^{-1}u - B\right)\left(1 + o\left(1\right)\right). \tag{10}$$

Using again (9) and then (10) we obtain a better estimate for $Q^{-1}\left(u\right)$:

$$\begin{aligned} Q^{-1}\left(u\right) &= \exp\left(A^{-1}u - B\right)\left(1 + o\left(\left(Q^{-1}\left(u\right)\right)^{-\theta}\right)\right) \\ &= \exp\left(A^{-1}u - B\right)\left(1 + o\left(\exp\left(-\theta A^{-1}u\right)\right)\right) \\ &= \frac{\kappa\gamma\alpha^{-\gamma/2}\beta^{-\frac{\beta}{1-\beta}}}{2}\exp\left(\frac{\gamma\left(1-\beta\right)}{2}u\right) + o\left(\exp\left(\frac{\gamma\left(1-\beta\right)\left(1-\theta\right)}{2}u\right)\right). \end{aligned}$$

Let us also check conditions (5)-(6). We obtain that $q'\left(\tau\right) = 2\gamma^{-1}/\tau_0$ if $\tau < \tau_0$ and $q'\left(\tau\right) = 2\gamma^{-1}/\tau$ if $\tau \geq \tau_0$, hence $\sigma_i^2\left(\tau\right) = 2\gamma^{-1}\beta^{2i}/\tau_0$ if $\beta^i\tau < \tau_0$ and $\sigma_i^2\left(\tau\right) = 2\gamma^{-1}\beta^i/\tau$ if $\beta^i\tau \geq \tau_0$. Thus

$$\sigma_\infty^2\left(\tau\right) = 2\gamma^{-1}\left(\sum_{j:\ \beta^j\tau\geq\tau_0}\beta^j/\tau + \sum_{j:\ \beta^j\tau<\tau_0}\beta^{2j}/\tau_0\right) \sim 2\gamma^{-1}\left(1-\beta\right)^{-1}/\tau,$$

so (5)-(6) are easily seen for the $c_i = \beta^i$ sequence, consequently Theorem 2 can be applied. It also follows that $\rho\left(\tau\right) = O\left(\tau^{1/2}\right)$ and $u_0 = 0$ in that Theorem. If we choose $\theta > 1/2$, we obtain

$$\bar{F}_{Y_t^*}(u) = \exp\left(-\log Q^{-1}(u) - \log \sigma_\infty\left(Q^{-1}(u)\right) - \int_0^u \left(Q^{-1}(v) + O\left(e^{\frac{\gamma(1-\beta)}{4}v}\right) dv\right)\right)$$

$$= \exp\left(-\frac{\kappa\alpha^{-\gamma/2}\beta^{-\frac{\beta}{1-\beta}}}{1-\beta} e^{\frac{\gamma(1-\beta)}{2}u} + O\left(e^{\frac{\gamma(1-\beta)}{4}u}\right)\right). \quad (11)$$

Taking into account that $Y_t^* \leq Y_t = \log\left(X_t^2\right)$, the lower bound is obtained for $\bar{F}_{X_t}(u)$ in (3).

To show the upper bound for the tail, let us first observe that, trivially, the increase of either α_+ or α_- does not make the tail of Y_t any lighter. Therefore, we can assume that $\alpha = \alpha_+ = \alpha_-$ and get an upper bound for the tail of this restricted model. In this case, let us introduce for each t a random variable $U_t^{**} \geq 0$ such that $U_t \leq U_t^{**}$ a.s. and $f_{U_t^{**}}(u) = K f_{U_t}(u)$ for all $u > 0$ with an appropriate $K > 0$. (Such a variable can easily be constructed.) Define also $h(y) = \beta y \chi_{\{y \geq 0\}} + \log(1 + \omega/\alpha)$. It follows from $\alpha_+ = \alpha_-$ that $h_i(y) \leq h(y)$ $(i = 1, 2)$ and thus it can be shown straightforwardly that $\bar{F}_{Y_t^{**}}(u) \geq \bar{F}_{Y_t}(u)$ holds for the stationary distribution of the model defined by

$$Y_t^{**} = h\left(Y_{t-1}^{**}\right) + U_t^{**}.$$

Indeed, let $\hat{Y}_0 = Y_0$ and define $\hat{Y}_t$ recursively as

$$\hat{Y}_t = h\left(\hat{Y}_{t-1}\right) + U_t^{**}.$$

Using $U_t \leq U_t^{**}$ we can prove by induction that $Y_t \leq \hat{Y}_t$:

$$Y_t \leq h\left(Y_{t-1}\right) + U_t \leq h\left(\hat{Y}_{t-1}\right) + U_t \leq h\left(\hat{Y}_{t-1}\right) + U_t^{**} = \hat{Y}_t.$$

Since the distribution of $\hat{Y}_t$ tends to the stationary distribution of Y_t^{**} as $t \to \infty$, Y_t is stochastically smaller than Y_t^{**}.

As $h(y) \geq 0$ for all y and $U_t^{**} \geq 0$ a.s., an alternative definition for Y_t^{**} is

$$Y_t^{**} = \beta Y_{t-1}^{**} + U_t^{**} + \log(1 + \omega/\alpha) = \sum_{i=0}^{\infty} \beta^i U_{t-i}^{**} + \frac{\log(1 + \omega/\alpha)}{1-\beta}. \quad (12)$$

The result of Klüppelberg and Lindner (2005) again gives that the tail of $\sum_{i=0}^{\infty} \beta^i U_{t-i}^{**}$ has the same form as the tail of Y_t^* (equation (11)). Completing the above reasoning with the effect of the last constant term in (12) it is easy to derive the upper bound for the tail of X_t in (3). $\square$

In some applications (e.g. Elek and Márkus (2008)) the $\alpha_- = 0$ restriction is used. Then the upper bound in (3) certainly holds and a slightly weaker lower bound can also be proven easily:

Proposition 1. *Assume the assumptions of Theorem 1 but allow $\alpha_- = 0$. Then for every $\delta > 0$ there exists a $K > 0$ such that*

$$\exp\left(-Ku^{(1+\delta)\gamma(1-\beta)}\right) \leq \bar{F}_{X_t}(u). \tag{13}$$

Proof. Let us use the same notations as in the proof of Theorem 1 and let Y_t^{***} be defined by

$$\begin{aligned} Y_t^{***} &= \beta Y_{t-1}^{***} + U_t^+ && \text{if} \quad Z_t \geq 0 \\ Y_t^{***} &= U_t^0 && \text{if} \quad Z_t < 0 \end{aligned}$$

where $U_t^0 = \log\left(\alpha_0 Z_t^2\right)$. It is easily shown that $Y_t^{***} \leq Y_t$ stochastically in this case. Moreover, as Z_t is symmetrically distributed, for every $n \in \mathrm{Z}^+$ with probability 2^{-n}

$$Y_t^{***} = \sum_{i=0}^{n-1} \beta^i U_{t-i}^+ + \beta^n Y_{t-n}^{***}.$$

Therefore, using the notations $q = \bar{F}_{Y_{t-n}^{***}}(0)$ and $Y_{t,n} = \sum_{i=0}^{n-1} \beta^i U_{t-i}^+$,

$$\bar{F}_{Y_t^{***}}(u) \geq q 2^{-n} \bar{F}_{Y_{t,n}}(u).$$

Moreover, similarly to the derivation of the tail of Y_t^*, it follows again from Theorem 2 that

$$\bar{F}_{Y_{t,n}}(u) = \exp\left(-Ke^{\frac{\gamma(1-\beta)}{2(1-\beta^n)}} + O\left(e^{\frac{\gamma(1-\beta)}{4(1-\beta^n)}}\right)\right)$$

with a suitable $K > 0$. Choosing n such that $1/\left(1-\beta^n\right) < 1+\delta$ we obtain

$$\bar{F}_{Y_t^{***}}(u) \geq \exp\left(-Ke^{\frac{(1+\delta)\gamma(1-\beta)}{2}}\right)$$

with a possibly different $K > 0$, and transforming Y_t to X_t gives the statement of the proposition. $\square$

3. Conclusions

In this paper we showed that the tail of a β-TARCH model can be approximated by Weibull-like distributions with exponent $\beta\,(1-\gamma)$ if the generating noise has Weibull-like tail with exponent γ. It is a natural question to ask how the tail behaviour is modified when AR- or MA-terms are added to the simple uncorrelated β-TARCH model. Unfortunately, this question is not yet settled but we conjecture that the more general ARMA-β-TARCH model has approximately a Weibull-like tail, too.

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