Megoldások: kismintás tulajdonságok

Elek Péter

1. feladat. *(Megoldás)*

a. $E(\widehat{p}_1) = p$ and $E(\widehat{p}_2) = 1/4 + p/2$, so only $\widehat{p}_1$ is unbiased.

b. $Var(\widehat{p}_1) = p(1-p)/4$ and $Var(\widehat{p}_2) = p(1-p)/16$.

c. $MSE(\widehat{p}_1) = p(1-p)/4$ because of unbiasedness, while

$$MSE(\widehat{p}_2) = Var(\widehat{p}_2) + Bias^2(\widehat{p}_2) = \left(3p^2 - 3p + 1\right)/16.$$

By solving the inequality we obtain that $MSE(\widehat{p}_2) < MSE(\widehat{p}_1)$ as long as $0.173 < p < 0.827$, i.e. when p is in the "middle" of the unit interval.

2. feladat. *(Megoldás)*

a. $E\left(\overline{X}\right) = E(X) = \theta$, so the sample mean is unbiased for θ. Its variance is $Var\left(\overline{X}\right) = Var(X)/n = \theta^2/n$.

b. The "survival function" of the exponential distribution with parameter θ is $\Pr(X_i > x) = \exp(-x/\theta)$ for all $x > 0$ (because of the formula for its cdf). Since H_n is the sample minimum and our sample is i.i.d., this implies $\Pr(H_n > x) = \prod_{i=1}^{n} \Pr(X_i > x) = \exp(-nx/\theta)$ for all $x > 0$. Hence $\Pr(nH_n > x) = \exp(-x/\theta)$ for all $x > 0$, which is just the survival function of an exponential distribution with parameter θ. Hence, $E(R_n) = \theta$ and thus R_n is also unbiased. Its variance is given by $Var(R_n) = \theta^2$.

c. Both estimators are unbiased but $Var\left(\overline{X}\right)/Var(R_n) = 1/n < 1$ if $n > 1$. (In fact $\overline{X}$ is much more efficient than R_n if n is large.) Hence we clearly prefer $\overline{X}$ over R_n.

3. feladat. *(Megoldás)*

See Amemiya Example 7.2.5.

Note that $Var\left(\overline{X}\right) \sim c_1/n$ as $n \to \infty$ (as usual in i.i.d. samples), while $Var(T_n) \sim c_2/n^2$, so the rate of decay is faster in the second case. This is a "non-regular" example. (For those who are interested: this "strange" behavior is the result of the fact that the support of the pdf (i.e. the set of x-s for which the pdf is positive) depends on the estimated parameter θ.))